

# NAHM ALGEBRAS

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ABSTRACT. Given a Lie algebra  $\mathfrak{g}$ , the *Nahm algebra* of  $\mathfrak{g}$  is the vector space  $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$  with the natural commutative, nonassociative algebra structure associated with the system of ordinary differential equations (1.1)-(1.3). Motivated by applications to these equations, we herein initiate the study of Nahm algebras.

## 1. INTRODUCTION

Let  $\mathfrak{g}$  be a real or complex Lie algebra. The *Nahm equations* for  $\mathfrak{g}$  are the following autonomous, first order differential equations:

$$(1.1) \quad \dot{x} = [y, z]$$

$$(1.2) \quad \dot{y} = [z, x]$$

$$(1.3) \quad \dot{z} = [x, y]$$

for  $x, y, z \in \mathfrak{g}$ . This system of equations is of interest in mathematical physics, especially in the case when  $\mathfrak{g}$  is a matrix Lie algebra. This is because certain types of solutions of (1.1)-(1.3) are equivalent to monopole solutions of the self-dual  $SU(2)$  Yang-Mills equations [2] [12]. Much work has been done to understand solutions of the Nahm equations in various physical and geometric contexts; good places to start for those interested are the papers [1] [2] [10] [12], and the references therein.

Let  $Q : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$  denote the mapping defined by the right hand side of the Nahm equations (1.1)-(1.3). This mapping is homogeneous quadratic, i.e.,  $Q(\alpha X) = \alpha^2 Q(X)$  for all  $\alpha \in \mathbb{K}$ ,  $X = (x_1, x_2, x_3) \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ . In 1960, L. Markus [11] noted that to every quadratic differential equation  $\dot{X} = Q(X)$  occurring in a vector space  $V$  over  $\mathbb{K}$ , there is associated a natural algebra. This algebra is  $\mathcal{A} = (V, \cdot)$  where the operation  $\cdot$  is the bilinearization of  $Q$  defined by

$$X \cdot Y = \frac{1}{2} Q^{(2)}(0)(X, Y),$$

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$X, Y \in V$ , where  $Q^{(2)}(0) : V \times V \rightarrow V$  is the second derivative of  $Q$  at 0. Clearly  $\mathcal{A}$  is a commutative algebra, but in general, it is nonassociative. We have  $Q(X) = X \cdot X$ , and if we abbreviate  $X^2 := X \cdot X$ , then we may write the differential equation as  $\dot{X} = X^2$ . Thus we may view quadratic differential equations as occurring in commutative, nonassociative algebras.

This algebraic perspective for quadratic differential equations is useful because the structure of the underlying algebra can give information about the trajectories (solution curves) of the differential equation. This is analogous to the situation with constant coefficient linear differential systems; such equations can be completely understood in terms of the theory of a vector space acted on by a single linear transformation. It is reasonable to expect that the theory of vector spaces with a bilinear mapping, i.e., algebras, would play a role in understanding quadratic differential equations. This line of investigation has been pursued by a number of authors. For surveys, see [6] or the monograph [15].

Applying these ideas to the Nahm equations (1.1)-(1.3) leads us to the following definition.

**Definition 1.1.** Let  $(\mathfrak{g}, [\cdot, \cdot])$  be a real or complex Lie algebra. The *Nahm algebra*  $(A(\mathfrak{g}), \cdot)$  associated to  $\mathfrak{g}$  is the vector space  $A(\mathfrak{g}) = \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$  with the multiplication

$$(1.4) \quad X \cdot Y = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \equiv \frac{1}{2} \begin{pmatrix} [x_2, y_3] + [y_2, x_3] \\ [x_3, y_1] + [y_3, x_1] \\ [x_1, y_2] + [y_1, x_2] \end{pmatrix}$$

for  $X = (x_1, x_2, x_3)^T, Y = (y_1, y_2, y_3)^T \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ .

In this paper, we initiate an investigation of Nahm algebras. As indicated, our eventual goal is to understand the Nahm equations. Thus throughout the paper, we will motivate the topics we discuss by referring to their relevance for quadratic differential equations occurring in commutative algebras. However, the paper itself is a purely algebraic study of Nahm algebras. The implications of our results for the Nahm equations will appear elsewhere.

Since our motivations lie in differential equations, all vector spaces and algebras herein are over the field  $\mathbb{K}$ , where  $\mathbb{K} = \mathbb{R}$  or  $\mathbb{K} = \mathbb{C}$ . However, many of the results hold for arbitrary fields of characteristic zero, and some hold in positive characteristic. Definition 1.1 shows our notation convention: lower case letters indicate elements of the Lie algebra  $\mathfrak{g}$ , and the corresponding upper case letters denote elements of the Nahm algebra  $A(\mathfrak{g})$ . We will frequently abbreviate the product in the Nahm algebra by concatenation  $XY = X \cdot Y$ . We will also use the following notation: for  $i = 1, 2, 3$ , we define the projection  $\pi_i : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$  by  $\pi_i(x_1, x_2, x_3)^t = x_i$ .

If  $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$  is a homomorphism of Lie algebras, then the mapping  $A(\phi) : A(\mathfrak{g}) \rightarrow A(\mathfrak{h})$  defined by  $A(\phi)(x_1, x_2, x_3)^t = (\phi(x_1), \phi(x_2), \phi(x_3))^t$  is clearly a homomorphism of the associated Nahm algebras. It follows that

the assignment  $\mathfrak{g} \rightsquigarrow A(\mathfrak{g})$  is a covariant functor from the category of Lie algebras to the category of Nahm algebras.

One might guess that the Jacobi identity in the Lie algebra  $\mathfrak{g}$  would lead to identities satisfied in the Nahm algebra  $A(\mathfrak{g})$ . Interestingly enough, this does not seem to be the case. For instance, the Nahm product (1.4) is not, in general, fourth power-associative, and thus Nahm algebras are not a subclass of some well-studied variety of commutative, power-associative algebras, such as Jordan algebras [14].

If  $T : \mathbb{K}^3 \rightarrow \mathbb{K}^3$  is a linear transformation, then  $T$  acts on  $A(\mathfrak{g}) = \mathfrak{g} \otimes \mathbb{K}^3$  in the obvious way:

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_1 \otimes T\mathbf{e}_1 + x_2 \otimes T\mathbf{e}_2 + x_3 \otimes T\mathbf{e}_3$$

where  $\mathbf{e}_i$  is the  $i$ th standard basis vector of  $\mathbb{K}^3$ . More specifically, if  $T$  is given by a  $3 \times 3$  matrix  $T = [T_{ij}]$  relative to the standard basis, then the action agrees with that obtained by formally multiplying the column vector  $(x_1, x_2, x_3)^t$  ( $x_i \in \mathfrak{g}$ ,  $i = 1, 2, 3$ ) on the left by the matrix  $T$ .

For any linear transformation  $L \in gl(\mathfrak{g})$ , we will denote the naturally induced transformation in  $gl(A(\mathfrak{g}))$  by  $\text{diag}(L)$ ; thus

$$\text{diag}(L) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} Lx_1 \\ Lx_2 \\ Lx_3 \end{pmatrix}$$

for  $X = (x_1, x_2, x_3)^T \in A(\mathfrak{g})$ .

For  $X = (x_1, x_2, x_3)^T \in A(\mathfrak{g})$ , the *left multiplication operator*  $L(X) \in gl(A(\mathfrak{g}))$  is defined by  $L(X)Y = XY$  for all  $Y \in A(\mathfrak{g})$ . Using (1.4), we see that  $L(X)$  has a block matrix representation given by

$$(1.5) \quad L(X) = \frac{1}{2} \begin{bmatrix} 0 & -\text{ad } x_3 & \text{ad } x_2 \\ \text{ad } x_3 & 0 & -\text{ad } x_1 \\ -\text{ad } x_2 & \text{ad } x_1 & 0 \end{bmatrix}$$

where for  $x \in \mathfrak{g}$ , the adjoint representation is given by  $(\text{ad } x)y = [x, y]$ .

This suggests the following definition. Let  $\rho : \mathfrak{g} \rightarrow gl(V)$  be a representation of  $\mathfrak{g}$  on a vector space  $V$ . For  $X \in A(\mathfrak{g})$ , we define an operator  $L_\rho(X) \in gl(V \times V \times V)$  as follows:

$$(1.6) \quad L_\rho(X) = \frac{1}{2} \begin{bmatrix} 0 & -\rho(x_3) & \rho(x_2) \\ \rho(x_3) & 0 & -\rho(x_1) \\ -\rho(x_2) & \rho(x_1) & 0 \end{bmatrix}.$$

Thus  $L_{\text{ad}}(X) = L(X)$ . We will use the operators  $L_\rho(X)$ ,  $X \in A(\mathfrak{g})$ , in our discussion of invariant bilinear forms.

We conclude this introduction with an outline of the sequel. In §2 we discuss the relationship between subalgebras and ideals of the Lie algebra  $\mathfrak{g}$  and subalgebras and ideals of the Nahm algebra  $A(\mathfrak{g})$ . In §3 we discuss how  $\mathbb{Z}_2$ -gradings of  $\mathfrak{g}$  naturally induce  $\mathbb{Z}_2$ -gradings of  $A(\mathfrak{g})$ . We also show that

every Nahm algebra  $A(\mathfrak{g})$  has a natural  $\mathbb{Z}_2$ -grading where the even subalgebra is a copy (as a vector space) of  $\mathfrak{g}$  itself. In §4 we discuss nilpotents of index 2 and idempotents in  $A(\mathfrak{g})$ . Roughly speaking, nilpotents in  $A(\mathfrak{g})$  are built from abelian subalgebras of  $\mathfrak{g}$ , and idempotents in  $A(\mathfrak{g})$  are built from subalgebras of  $\mathfrak{g}$  which are isomorphic to  $so(3, \mathbb{K})$ . In §5, we discuss simplicity and prove that the Lie algebra  $\mathfrak{g}$  is simple if and only if the Nahm algebra  $A(\mathfrak{g})$  is simple. We also give an example to show that simple Nahm algebras can have simple subalgebras which are not themselves Nahm algebras of a Lie algebra. We then turn to semisimplicity and prove that the Lie algebra  $\mathfrak{g}$  is semisimple if and only if the Nahm algebra  $A(\mathfrak{g})$  is semisimple. In §6 we show that the radical of a Nahm algebra is the Nahm algebra of the radical of the Lie algebra. It follows from this that every Nahm algebra has a Levi-Malcev decomposition. In §7 we consider invariant bilinear forms for Nahm algebras. Any representation of  $\mathfrak{g}$  naturally induces an invariant trace form on  $A(\mathfrak{g})$ . The form so induced by the adjoint representation, which we call the standard form, measures the semisimplicity of  $A(\mathfrak{g})$  in exact analogy with the role of the Killing form on  $\mathfrak{g}$  itself:  $A(\mathfrak{g})$  is semisimple if and only if its standard form is nondegenerate. In §8 we consider derivations of  $A(\mathfrak{g})$ . We show that the derivation algebra of any Nahm algebra has two natural subalgebras: one is a copy of  $\text{ad}(\mathfrak{g})$ , and the other is a copy of  $so(3, \mathbb{K})$ . For  $A(\mathfrak{g})$  simple, we prove that the derivation algebra is exactly the direct sum of these two subalgebras. We first show this for  $\mathbb{K} = \mathbb{C}$  and then note that the result follows in the real case by the invariance of dimension of derivation algebras. Along the way, we also prove a version of Schur's lemma for complex, simple Nahm algebras. Finally, in §9 we discuss automorphisms of Nahm algebras, and we characterize the automorphism group of a simple Nahm algebra: it is a direct product of the automorphism group of the Lie algebra and  $SO(3, \mathbb{K})$ .

We should mention that for any anticommutative algebra  $\mathfrak{g}$ , one can certainly define its “Nahm algebra”  $A(\mathfrak{g})$  as in Definition 1.1. Indeed, some of what follows is valid in the case where, for example,  $\mathfrak{g}$  is a Malcev algebra. However, in this paper  $\mathfrak{g}$  will be a Lie algebra, and the structure of the Nahm algebra will turn out to be closely related to the structure of  $\mathfrak{g}$  itself.

## 2. SUBALGEBRAS AND IDEALS

A *subalgebra* of an algebra  $\mathcal{A}$  is a subspace  $\mathcal{B}$  such that  $\mathcal{B}^2 \subseteq \mathcal{B}$ , that is,  $XY \in \mathcal{B}$  for all  $X, Y \in \mathcal{B}$ . Given a fixed  $P \in \mathcal{A}$ , the *subalgebra generated by*  $P$  is defined by

$$(2.1) \quad \langle P \rangle = \mathbb{R}[P],$$

which is the set of all polynomials in  $P$ . For a commutative algebra  $\mathcal{A}$  with its associated quadratic differential equation  $\dot{X} = X^2$ , the unique solution  $X(t; P)$  to the equation satisfying the initial condition  $X(0) = P$  remains in the subalgebra  $\langle P \rangle$ , i.e.,  $X(t; P) \in \langle P \rangle$  for all  $t$  [11] [6] [15].

For a Nahm algebra  $A = A(\mathfrak{g})$  of a Lie algebra  $\mathfrak{g}$ , it is reasonable to expect that subalgebras in  $A$  are related to subalgebras in  $\mathfrak{g}$ .

**Theorem 2.1.** *Let  $\mathfrak{m}_i \subseteq \mathfrak{g}$ ,  $i = 1, 2, 3$ , be subspaces, and let  $M = \mathfrak{m}_1 \times \mathfrak{m}_2 \times \mathfrak{m}_3$ . Then  $M$  is a subalgebra of  $A(\mathfrak{g})$  if and only if  $[\mathfrak{m}_i, \mathfrak{m}_{i+1}] \subseteq \mathfrak{m}_{i+2}$  for  $i = 1, 2, 3$  (where index addition is modulo 3).*

*Proof.* This follows immediately from considering the components of the product  $XY$  for  $X, Y \in M$ . ■

*Remark 2.2.* It is important to note that Theorem 2.1 does not characterize arbitrary subalgebras of  $A(\mathfrak{g})$ ; it only characterizes those which are direct products of subspaces in  $\mathfrak{g}$ .

**Corollary 2.3.** *Let  $\mathfrak{m} \subseteq \mathfrak{g}$  be a subspace. Then  $M = \mathfrak{m} \times \mathfrak{m} \times \mathfrak{m}$  is a subalgebra of  $A$  if and only if  $\mathfrak{m}$  is a subalgebra of  $\mathfrak{g}$ . In this case,  $M = A(\mathfrak{m})$  is the Nahm algebra of  $\mathfrak{m}$ .*

An *ideal* of an algebra  $\mathcal{A}$  is a subspace  $\mathcal{J}$  such that  $\mathcal{J}\mathcal{A} = \mathcal{A}\mathcal{J} \subseteq \mathcal{J}$ , that is,  $XY \in \mathcal{J}$  and  $YX \in \mathcal{J}$  for all  $X \in \mathcal{A}, Y \in \mathcal{J}$ . (Since we are dealing only with commutative and Lie algebras here, “ideal” for us means “two-sided ideal”.) In case  $\mathcal{A}$  is a commutative algebra, the presence of an ideal  $\mathcal{J}$  in  $\mathcal{A}$  implies that the associated quadratic differential equation can be decoupled into a quadratic equation in the quotient space  $\mathcal{A}/\mathcal{J}$  and a (nonhomogeneous) quadratic differential equation in  $\mathcal{J}$ ; see [15], p.23.

Let  $A(\mathfrak{g})$  be the Nahm algebra of a Lie algebra  $\mathfrak{g}$ . As one might expect, ideals in  $A(\mathfrak{g})$  are closely related to ideals in  $\mathfrak{g}$ .

**Theorem 2.4.** *Let  $J$  be an ideal of  $A(\mathfrak{g})$ , and let  $\mathfrak{h}_i = \pi_i(J) \subseteq \mathfrak{g}$ . Then  $[\mathfrak{g}, \mathfrak{h}_i] \subseteq \mathfrak{h}_{i+1} \cap \mathfrak{h}_{i+2}$  for  $i = 1, 2, 3$  (where index addition is modulo 3).*

*Proof.* Fix  $y_1 \in \mathfrak{h}_1$ . There exists  $y_j \in \mathfrak{h}_j$ ,  $j = 2, 3$ , such that  $(y_1, y_2, y_3)^t \in J$ . For all  $x \in \mathfrak{g}$ ,

$$\begin{pmatrix} 0 \\ x \\ x \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} [x, y_3] + [y_2, x] \\ [x, y_1] \\ [y_1, x] \end{pmatrix} \in J.$$

Thus  $[x, y_1] \in \mathfrak{h}_2 \cap \mathfrak{h}_3$ . Similar calculations show the other inclusions. ■

**Corollary 2.5.** *Let  $J$  be an ideal of  $A(\mathfrak{g})$  and let  $\mathfrak{h}_i = \pi_i(J)$ ,  $i = 1, 2, 3$ . then  $\mathfrak{h}_1 \cap \mathfrak{h}_2 \cap \mathfrak{h}_3$  is an ideal of  $\mathfrak{g}$ .*

**Theorem 2.6.** *Let  $\mathfrak{h}_i \subseteq \mathfrak{g}$ ,  $i = 1, 2, 3$ , be subspaces, and let  $J = \mathfrak{h}_1 \times \mathfrak{h}_2 \times \mathfrak{h}_3$ . Then  $J$  is an ideal of  $A(\mathfrak{g})$  if and only if  $[\mathfrak{g}, \mathfrak{h}_i] \subseteq \mathfrak{h}_{i+1} \cap \mathfrak{h}_{i+2}$  for  $i = 1, 2, 3$  (where index addition is modulo 3).*

*Proof.* The necessity of the stated condition is Theorem 2.4, while the sufficiency is clear from (1.4). ■

**Corollary 2.7.** *Let  $\mathfrak{h} \subseteq \mathfrak{g}$  be a subspace. Then  $\mathfrak{h} \times \mathfrak{h} \times \mathfrak{h}$  is an ideal of  $A(\mathfrak{g})$  if and only if  $\mathfrak{h}$  is an ideal of  $\mathfrak{g}$ .*

3.  $\mathbb{Z}_2$ -GRADINGS

When a Lie algebra  $\mathfrak{g}$  has a  $\mathbb{Z}_2$ -grading, it induces an interesting class of subalgebras of its associated Nahm algebra  $A(\mathfrak{g})$ . Thus assume  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  where  $\mathfrak{g}_0$  is a subalgebra,  $\mathfrak{g}_1$  is a subspace,  $[\mathfrak{g}_0, \mathfrak{g}_1] \subseteq \mathfrak{g}_1$  and  $[\mathfrak{g}_1, \mathfrak{g}_1] \subseteq \mathfrak{g}_0$ . (For example,  $\mathfrak{g}$  could be a semisimple Lie algebra with Cartan decomposition  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ .) Let

$$(3.1) \quad A_{011} = \mathfrak{g}_0 \times \mathfrak{g}_1 \times \mathfrak{g}_1$$

and similarly define  $A_{101}$  and  $A_{110}$ . By Theorem 2.1,  $A_{011}$ ,  $A_{101}$  and  $A_{110}$  are subalgebras of  $A(\mathfrak{g})$ . Let

$$(3.2) \quad A_{100} = \mathfrak{g}_1 \times \mathfrak{g}_0 \times \mathfrak{g}_0$$

and similarly define  $A_{010}$  and  $A_{001}$ . Then the following properties are easily seen to hold:

$$(3.3) \quad A(\mathfrak{g}) = A_{011} \oplus A_{100}$$

$$(3.4) \quad A_{011} \cdot A_{100} \subseteq A_{100}$$

$$(3.5) \quad A_{100} \cdot A_{100} \subseteq A_{011}.$$

Similar results hold for the other subalgebras and subspaces. Therefore the  $\mathbb{Z}_2$ -grading  $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$  naturally induces three  $\mathbb{Z}_2$ -gradings of the Nahm algebra  $A(\mathfrak{g})$ . If a commutative algebra  $\mathcal{A}$  has a  $\mathbb{Z}_2$ -grading  $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ , then the differential equation  $\dot{X} = X^2$  in  $\mathcal{A}$  can be decomposed relative to the grading, and this can give information about the trajectories [4].

Every Nahm algebra carries a natural  $\mathbb{Z}_2$ -grading whether the underlying Lie algebra is  $\mathbb{Z}_2$ -graded or not. For each  $x \in \mathfrak{g}$ , let

$$(3.6) \quad \Delta(x) = \begin{pmatrix} x \\ x \\ x \end{pmatrix}$$

and let

$$(3.7) \quad \Delta(\mathfrak{g}) = \{\Delta(x) : x \in \mathfrak{g}\}.$$

As a subspace,  $\Delta(\mathfrak{g})$  is just a copy of  $\mathfrak{g}$  itself. Let

$$W(\mathfrak{g}) = \{X \in A(\mathfrak{g}) : x_1 + x_2 + x_3 = 0\}.$$

Then the following properties hold.

**Theorem 3.1.** 1.  $\Delta(\mathfrak{g})$  is an abelian subalgebra of  $A(\mathfrak{g})$ , i.e.,

$$\Delta(x)\Delta(y) = 0$$

for all  $x, y \in \mathfrak{g}$ .

2.  $A(\mathfrak{g}) = \Delta(\mathfrak{g}) \oplus W(\mathfrak{g})$ .
3.  $\Delta(\mathfrak{g}) \cdot W(\mathfrak{g}) \subseteq W(\mathfrak{g})$ .
4.  $W(\mathfrak{g}) \cdot W(\mathfrak{g}) \subseteq \Delta(\mathfrak{g})$ .

*Proof.* 1. This is clear from (1.4).

2. If  $\Delta(x) \in \Delta(\mathfrak{g}) \cap W(\mathfrak{g})$ , then  $3x = 0$  and thus  $\Delta(x) = 0$ . For  $X \in A(\mathfrak{g})$ , define mappings  $P_\Delta : A(\mathfrak{g}) \rightarrow \Delta(\mathfrak{g})$  and  $P_W : A(\mathfrak{g}) \rightarrow W(\mathfrak{g})$  by

$$(3.8) \quad P_\Delta(X) = \Delta\left(\frac{1}{3}(x_1 + x_2 + x_3)\right)$$

and

$$(3.9) \quad P_W(X) = \frac{1}{3} \begin{pmatrix} 2x_1 - x_2 - x_3 \\ -x_1 + 2x_2 - x_3 \\ -x_1 - x_2 + 2x_3 \end{pmatrix}.$$

Then  $P_\Delta^2 = P_\Delta$ ,  $P_W^2 = P_W$ , and  $X = P_\Delta(X) + P_W(X)$  for each  $X \in A(\mathfrak{g})$ , which proves the desired result and also shows that  $P_\Delta$  and  $P_W$  are the projectors onto  $\Delta(\mathfrak{g})$  and  $W(\mathfrak{g})$ , respectively.

3. For  $x \in \mathfrak{g}$ ,  $Y \in W(\mathfrak{g})$ , we have

$$\Delta(x)Y = \frac{1}{2} \begin{pmatrix} [x, y_3 - y_2] \\ [x, y_1 - y_3] \\ [x, y_2 - y_1] \end{pmatrix}.$$

Summing up entries shows  $\Delta(x)Y \in W(\mathfrak{g})$ .

4. Fix  $X, Y \in W(\mathfrak{g})$ , and let  $s_{12}$  denote the difference between the first and second entries of the product  $2XY$ . Then

$$\begin{aligned} s_{12} &= ([x_2, y_3] + [y_2, x_3]) - ([x_3, y_1] + [y_3, x_1]) \\ &= [x_1 + x_2, y_3] + [y_1 + y_2, x_3] \\ &= [x_1 + x_2, y_3] + [y_1 + y_2, x_3], \end{aligned}$$

using anticommutativity of the Lie bracket. Adding the quantity  $[x_3, y_3] + [y_3, x_3] = 0$ , we obtain

$$\begin{aligned} s_{12} &= [x_1 + x_2 + x_3, y_3] + [y_1 + y_2 + y_3, x_3] \\ &= 0, \end{aligned}$$

since  $X, Y \in W(\mathfrak{g})$ . Similarly, we can show that the difference between the second and third entries of  $2XY$  is 0. Thus all three entries are equal, which proves that  $XY \in \Delta(\mathfrak{g})$ . ■

**Corollary 3.2.** *The decomposition  $A(\mathfrak{g}) = \Delta(\mathfrak{g}) \oplus W(\mathfrak{g})$  is a  $\mathbb{Z}_2$ -grading for  $A(\mathfrak{g})$  with even subalgebra  $\Delta(\mathfrak{g})$  and odd subspace  $W(\mathfrak{g})$ .*

**Definition 3.3.**  $\Delta(\mathfrak{g}) \subset A(\mathfrak{g})$  is called the *diagonal subalgebra* of  $A(\mathfrak{g})$ .

The decomposition  $A(\mathfrak{g}) = \Delta(\mathfrak{g}) \oplus W(\mathfrak{g})$  of a Nahm algebra has implications for the Nahm equations as we will discuss elsewhere.

## 4. IDEMPOTENTS AND NILPOTENTS

A *nilpotent* element  $N$  (of index 2) of an algebra  $\mathcal{A}$  is one which satisfies  $N^2 = 0$ . If  $\mathcal{A}$  is the commutative algebra associated to a quadratic differential equation  $\dot{X} = X^2$ , then nilpotents correspond to nonzero equilibria, i.e., stationary points. In particular, if  $N$  is a nilpotent, then the constant function  $X(t) = N$  is a solution:

$$N^2 = X^2 = \frac{dX}{dt} = \frac{dN}{dt} = 0.$$

Conversely, the same calculation shows that a constant function  $X(t) = N$  is a solution only if  $N^2 = 0$ .

Now let  $\mathfrak{g}$  be a Lie algebra with Nahm algebra  $A = A(\mathfrak{g})$ . Assume that  $N = (n_1, n_2, n_3)^t \in A$  is a nilpotent. Since  $N^2 = 0$ , we have  $[n_i, n_j] = 0$  for all  $i, j$ . Thus the Lie subalgebra  $\langle n_1, n_2, n_3 \rangle$  of  $\mathfrak{g}$  generated by the set  $\{n_1, n_2, n_3\}$  is abelian, and the subspace  $\mathbb{K} \cdot n_1 \times \mathbb{K} \cdot n_2 \times \mathbb{K} \cdot n_3 \subseteq A$  is an abelian subalgebra. Conversely, given any abelian subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ , any three-element set  $\{n_1, n_2, n_3\} \subset \mathfrak{h}$  gives an abelian subalgebra  $\mathbb{K} \cdot n_1 \times \mathbb{K} \cdot n_2 \times \mathbb{K} \cdot n_3$  of  $A$ , every element of which is a nilpotent. Indeed, if  $B$  is an abelian subalgebra of  $A$ , then clearly every element of  $B$  is a nilpotent. This applies, for instance, to the diagonal subalgebra  $\Delta(\mathfrak{g})$ .

An *idempotent* of an algebra  $\mathcal{A}$  is a nonzero element  $E \in \mathcal{A}$  satisfying  $E^2 = E$ . If  $\mathcal{A}$  is the commutative algebra associated to a quadratic differential equation  $\dot{X} = X^2$ , then idempotents gives solutions which blow up in finite time along rays. For arbitrary  $E \in \mathcal{A}$  and for  $a \neq 0$ , the function  $X(t) = aE/(1-at)$  blows up in finite time at  $t = 1/a$ . If  $E$  is an idempotent, then  $X(t)$  is the solution to the differential equation with initial value  $aE$ :

$$\frac{a^2 E^2}{(1-at)^2} = X^2 = \frac{dX}{dt} = \frac{a^2 E}{(1-at)^2}.$$

Conversely, the same calculation shows that  $X(t)$  is a solution only if  $E = E^2$  is an idempotent.

Let  $\mathfrak{g}$  be a Lie algebra with Nahm algebra  $A = A(\mathfrak{g})$ . Assume that  $E = (e_1, e_2, e_3)^t \in A(\mathfrak{g})$  is an idempotent. Since  $E = E^2$ , we have

$$(4.1) \quad \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} [e_2, e_3] \\ [e_3, e_1] \\ [e_1, e_2] \end{pmatrix}.$$

If any  $e_i = 0$ , then (4.1) shows that all  $e_i = 0$ , which contradicts  $E$  being an idempotent. Suppose  $\{e_1, e_2, e_3\}$  is linearly dependent, say,  $e_3 = ae_1 + be_2$  for some  $a, b \in \mathbb{K}$ . Then

$$e_1 = [e_2, ae_1 + be_2] = -a[e_1, e_2] = -ae_3 = -a^2e_1 - abe_2,$$

and

$$e_2 = [ae_1 + be_2, e_1] = -b[e_1, e_2] = -be_3 = -abe_1 - b^2e_2.$$



Thus  $a^2 = b^2 = -1$  and  $ab = 0$ , which is a contradiction. Therefore  $\{e_1, e_2, e_3\}$  is linearly independent, and (4.1) shows that it satisfies the defining relations of the Lie algebra  $so(3, \mathbb{K})$  (or isomorphically, the Lie algebra  $\mathbb{K}^3$  with the cross-product as Lie bracket).

Conversely, assume that  $\mathfrak{g}$  contains a subalgebra  $B$  isomorphic to  $so(3, \mathbb{K})$ . Let  $\{e_1, e_2, e_3\}$  be an ordered basis for  $B$  satisfying the relations  $[e_i, e_{i+1}] = e_{i+2}$  (where index addition is modulo 3). Set  $E = (e_1, e_2, e_3)^t$ . Then  $E$  is an idempotent.

## 5. SIMPLE AND SEMISIMPLE ALGEBRAS

An algebra  $\mathcal{A}$  is called *simple* if  $\mathcal{A}^2 \neq \{0\}$  and  $\mathcal{A}$  contains no nontrivial ideals, that is, the only nonzero ideal of  $\mathcal{A}$  is  $\mathcal{A}$  itself. We now consider the relationship between simplicity of a Lie algebra  $\mathfrak{g}$  and the simplicity of its Nahm algebra  $A(\mathfrak{g})$ .

**Theorem 5.1.**  *$A(\mathfrak{g})$  is simple if and only if  $\mathfrak{g}$  is simple.*

*Proof.* Assume  $A(\mathfrak{g})$  is simple and let  $\mathfrak{h}$  be a nonzero ideal of  $\mathfrak{g}$ . By Corollary 2.7,  $\mathfrak{h} \times \mathfrak{h} \times \mathfrak{h}$  is an ideal of  $A(\mathfrak{g})$ . Thus  $A(\mathfrak{g}) = \mathfrak{h} \times \mathfrak{h} \times \mathfrak{h}$ , which implies  $\mathfrak{h} = \mathfrak{g}$ . Conversely, assume  $\mathfrak{g}$  is simple, let  $J$  be an ideal of  $A(\mathfrak{g})$ , and let  $\mathfrak{h}_i = \pi_i(J)$ . By Theorem 2.4,  $[\mathfrak{g}, [\mathfrak{g}, \mathfrak{h}_i]] \subseteq \mathfrak{h}_1 \cap \mathfrak{h}_2 \cap \mathfrak{h}_3$  for  $i = 1, 2, 3$ . By Corollary 2.5,  $\mathfrak{h}_1 \cap \mathfrak{h}_2 \cap \mathfrak{h}_3$  is an ideal. If  $\mathfrak{h}_1 \cap \mathfrak{h}_2 \cap \mathfrak{h}_3 = \mathfrak{g}$ , then  $J = \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} = A(\mathfrak{g})$ , and thus we may assume  $\mathfrak{h}_1 \cap \mathfrak{h}_2 \cap \mathfrak{h}_3 = \{0\}$  since  $\mathfrak{g}$  is simple. Now for  $x, y \in \mathfrak{g}$ ,  $z \in \mathfrak{h}_i$ , the Jacobi identity gives  $[[x, y], z] = [[x, z], y] + [y, [x, z]] \in [\mathfrak{g}, [\mathfrak{g}, \mathfrak{h}_i]] = \{0\}$ . Therefore  $[[\mathfrak{g}, \mathfrak{g}], \mathfrak{h}_i] = \{0\}$ . But  $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$  since  $\mathfrak{g}$  is simple, and thus  $[\mathfrak{g}, \mathfrak{h}_i] = \{0\}$ . Hence each  $\mathfrak{h}_i$  is contained in the center of  $\mathfrak{g}$ , and thus each  $\mathfrak{h}_i = \{0\}$ . Therefore  $J = \{0\}$ , which shows that  $A(\mathfrak{g})$  is simple. ■

The next example shows that simple Nahm algebras can have simple subalgebras which are not themselves Nahm algebras of a Lie algebra.

**Example 5.2.** Let  $\mathfrak{g} = \mathbb{R}^3$  with the bracket being the cross-product. (Thus  $\mathfrak{g}$  is isomorphic to  $so(3)$ .) Then  $\mathfrak{g}$  is simple, and thus the Nahm algebra  $A(\mathfrak{g}) = \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3$  is simple by Theorem 5.1. Now let  $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$  denote the standard basis of  $\mathbb{R}^3$ , and consider the subspace  $B = \mathbb{R} \cdot \mathbf{e}_1 \times \mathbb{R} \cdot \mathbf{e}_2 \times \mathbb{R} \cdot \mathbf{e}_3 \subseteq A(\mathfrak{g})$ . Then  $B$  is clearly a subalgebra, but it is not the Nahm algebra of a subalgebra of  $\mathfrak{g}$  (or of any Lie algebra, for that matter). In addition,  $B$  is simple as the following argument shows. Let  $J \subseteq B$  be a nonzero ideal, and assume  $\mathbf{0} \neq (a\mathbf{e}_1, b\mathbf{e}_2, c\mathbf{e}_3)^t \in J$ . Assume first that  $a \neq 0$ ,  $b = c = 0$ . Then

$$(\mathbf{0}, \mathbf{e}_2, \mathbf{0})^t \cdot (a\mathbf{e}_1, \mathbf{0}, \mathbf{0})^t = \frac{1}{2}(\mathbf{0}, \mathbf{0}, a\mathbf{e}_3)^t \in J$$

and

$$(\mathbf{0}, \mathbf{0}, \mathbf{e}_3)^t \cdot (a\mathbf{e}_1, \mathbf{0}, \mathbf{0})^t = \frac{1}{2}(\mathbf{0}, a\mathbf{e}_2, \mathbf{0})^t \in J.$$

This shows  $J = B$ . Applying similar arguments shows that we may assume that at least two of  $a, b, c$  are nonzero. Thus assume  $a = 0, b, c \neq 0$ . Then

$$(\mathbf{0}, \mathbf{e}_2, \mathbf{0})^t \cdot (\mathbf{0}, b\mathbf{e}_2, c\mathbf{e}_3)^t = \frac{1}{2}(c\mathbf{e}_1, \mathbf{0}, \mathbf{0})^t \in J,$$

and repeating the argument above gives  $J = B$ . Applying similar arguments shows that we may assume that each of  $a, b, c$  are nonzero. Then

$$(\mathbf{0}, b\mathbf{e}_2, -c\mathbf{e}_3)^t \cdot (a\mathbf{e}_1, b\mathbf{e}_2, c\mathbf{e}_3)^t = \frac{1}{2}(\mathbf{0}, -cae_2, abe_3)^t \in J,$$

and repeating the preceding arguments gives  $J = B$ . Thus  $B$  is simple, as claimed.

An algebra  $\mathcal{A}$  is called *semisimple* if there exist ideals  $\mathcal{A}_1, \dots, \mathcal{A}_n$ , each of which is a simple algebra, such that

$$(5.1) \quad \mathcal{A} = \mathcal{A}_1 \oplus \dots \oplus \mathcal{A}_n,$$

a direct sum of subalgebras. Any ideal  $\mathcal{J}$  of  $\mathcal{A}$  is given by a direct sum  $\mathcal{J} = \mathcal{A}_{i_1} \oplus \dots \oplus \mathcal{A}_{i_k}$  for suitable  $\mathcal{A}_{i_j}$ . In particular,

$$(5.2) \quad \mathcal{J}^2 = \mathcal{A}_{i_1}^2 \oplus \dots \oplus \mathcal{A}_{i_k}^2 = \mathcal{A}_{i_1} \oplus \dots \oplus \mathcal{A}_{i_k} = \mathcal{J}$$

using  $\mathcal{A}_p \mathcal{A}_q = 0$  if  $p \neq q$  and  $\mathcal{A}_p^2 = \mathcal{A}_p$ .

If  $\mathcal{A}$  is a semisimple commutative algebra, then the associated quadratic differential equation  $\dot{X} = X^2$  in  $\mathcal{A}$  can be decoupled into differential equations occurring in the simple subalgebras  $\mathcal{A}_j$ . This follows from the remarks above about ideals (see [15], p.23), but can be just as easily seen directly. Thus if  $X(t)$  is a solution and  $X = X_1 + \dots + X_n$  is the decomposition of  $X$  given by (5.1), then

$$\dot{X}_1 + \dots + \dot{X}_n = \dot{X} = X^2 = (X_1 + \dots + X_n)^2 = X_1^2 + \dots + X_n^2,$$

since  $X_i X_j = 0$  for  $i \neq j$ . Thus  $\dot{X}_j = X_j^2$  for  $j = 1, \dots, n$ . See [6] [15].

In analogy with our discussion of simplicity, we now show the relationship between the semisimplicity of a Lie algebra  $\mathfrak{g}$  and the semisimplicity of its associated Nahm algebra  $A(\mathfrak{g})$ .

**Theorem 5.3.**  *$A(\mathfrak{g})$  is semisimple if and only if  $\mathfrak{g}$  is semisimple.*

*Proof.* Let  $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_n$  be semisimple with each  $\mathfrak{g}_i$  simple. By Corollary 2.7 and Theorem 5.1, each  $\mathfrak{g}_i \times \mathfrak{g}_i \times \mathfrak{g}_i$  is an ideal of  $A(\mathfrak{g})$  and a simple algebra. Since  $A(\mathfrak{g}) = \bigoplus_{i=1}^n (\mathfrak{g}_i \times \mathfrak{g}_i \times \mathfrak{g}_i)$ , a direct sum,  $A(\mathfrak{g})$  is semisimple.

Conversely, assume  $A(\mathfrak{g}) = A_1 \oplus \dots \oplus A_n$  is semisimple with each  $A_i$  simple. Let  $\mathfrak{h} \neq \{0\}$  be a solvable ideal of  $\mathfrak{g}$ . Then

$$\mathfrak{h} \supset \mathfrak{h}^{(2)} \supset \dots \supset \mathfrak{h}^{(N)} = \{0\}$$

for some  $N \geq 2$ , where  $\mathfrak{h}^{(1)} = \mathfrak{h}$ ,  $\mathfrak{h}^{(k)} = [\mathfrak{h}^{(k-1)}, \mathfrak{h}^{(k-1)}]$  for  $k > 1$ , and each containment  $\supset$  is proper. From Corollary 2.7,  $J = \mathfrak{h} \times \mathfrak{h} \times \mathfrak{h}$  is an ideal of  $A(\mathfrak{g})$ . Now

$$J^2 = \mathfrak{h}^{(2)} \times \mathfrak{h}^{(2)} \times \mathfrak{h}^{(2)} \subset \mathfrak{h} \times \mathfrak{h} \times \mathfrak{h} = J,$$

is a proper containment, contradicting (5.2). Thus  $\mathfrak{g}$  has no nonzero solvable ideals, which implies that  $\mathfrak{g}$  is semisimple. ■

## 6. THE RADICAL; LEVI-MALCEV DECOMPOSITIONS

The *radical* of an algebra  $\mathcal{A}$ , denoted by  $\text{rad } \mathcal{A}$ , is an ideal of  $\mathcal{A}$  characterized as follows: if  $\mathcal{J}$  is an ideal of  $\mathcal{A}$  and  $\mathcal{A}/\mathcal{J}$  is semisimple, then  $\text{rad } \mathcal{A} \subseteq \mathcal{J}$ . The existence of  $\text{rad } \mathcal{A}$  can be shown using the Chinese Remainder Theorem; see Walcher [15], pp. 3-7. The radical of a Lie algebra can also be defined as being its maximal solvable ideal of  $\mathfrak{g}$  [5] [13]. For an algebra  $\mathcal{A}$ , if there exists a semisimple subalgebra  $\mathcal{S}$  such that  $\mathcal{A} = \mathcal{S} \oplus \text{rad } \mathcal{A}$ , then  $\mathcal{A}$  is said to have a *Levi-Malcev decomposition* and  $\mathcal{S}$  is called a *Levi factor*. For instance, every Lie algebra has a Levi-Malcev decomposition [5]. For a commutative algebra with a Levi-Malcev decomposition, the associated quadratic differential equation can be decoupled into an equation in the Levi factor and a nonautonomous equation in the radical [6].

The relationship between the radical of a Lie algebra and the radical of its Nahm algebra is contained in the following result.

**Theorem 6.1.**  $\text{rad } A(\mathfrak{g}) = A(\text{rad } \mathfrak{g})$ .

*Proof.* First observe that

$$(6.1) \quad A(\mathfrak{g})/A(\text{rad } \mathfrak{g}) \cong A(\mathfrak{g}/\text{rad } \mathfrak{g})$$

by a standard isomorphism theorem for vector spaces. Thus  $A(\mathfrak{g})/A(\text{rad } \mathfrak{g})$  is semisimple by Theorem 5.3. Now suppose  $J \subseteq A(\text{rad } \mathfrak{g})$  is an ideal of  $\mathfrak{g}$  such that  $A(\mathfrak{g})/J$  is semisimple. Then  $A(\text{rad } \mathfrak{g})/J$  is an ideal of  $A(\mathfrak{g})/J$ . Now  $A(\text{rad } \mathfrak{g})$  is a solvable ideal of  $A(\mathfrak{g})$  since  $\text{rad } \mathfrak{g}$  is a solvable ideal of  $\mathfrak{g}$ . This implies  $A(\text{rad } \mathfrak{g})/J$  is a solvable ideal of  $A(\mathfrak{g})/J$ . Since  $A(\mathfrak{g})/J$  is semisimple, we must have  $J = A(\text{rad } \mathfrak{g})$ . This completes the proof. ■

Since every Lie algebra  $\mathfrak{g}$  has a Levi-Malcev decomposition  $\mathfrak{g} = \mathfrak{s} \oplus \text{rad } \mathfrak{g}$  where  $\mathfrak{s}$  is semisimple, we immediately obtain the following result for Nahm algebras.

**Corollary 6.2.** *Every Nahm algebra has a Levi-Malcev decomposition.*

*Proof.* Let  $\mathfrak{g} = \mathfrak{s} \oplus \text{rad } \mathfrak{g}$  be a Levi-Malcev decomposition with  $\mathfrak{s}$  a semisimple Levi factor. Then  $A(\mathfrak{g}) = A(\mathfrak{s}) \oplus A(\text{rad } \mathfrak{g})$ . By Theorem 5.3,  $A(\mathfrak{s})$  is semisimple. By Theorem 6.1,  $A(\text{rad } \mathfrak{g}) = \text{rad } A(\mathfrak{g})$ . This completes the proof. ■

## 7. INVARIANT BILINEAR FORMS

For many interesting classes of algebras  $\mathcal{A}$ , there exists a bilinear form  $F : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{K}$  which reflects the structure of the algebra. The forms of interest are those which are *invariant* (or *associative*), meaning

$$F(X \cdot Y, Z) = F(X, Y \cdot Z)$$

for all  $X, Y, Z \in \mathcal{A}$ . The *radical* of a bilinear form  $F : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{K}$  on an algebra  $\mathcal{A}$  is the subspace  $\text{rad } F = \{X \in \mathcal{A} : F(X, Y) = 0 \text{ for all } Y \in \mathcal{A}\}$ . If  $F$  is invariant, then the radical is an ideal: for  $Z \in \text{rad } F$  and for all  $X, Y \in \mathcal{A}$ ,  $F(Y, X \cdot Z) = F(Y \cdot X, Z) = 0$ , which implies  $X \cdot Z \in \text{rad } F$ . A bilinear form  $F : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{K}$  is *nondegenerate* if its radical is the zero subspace, i.e.,  $\text{rad } F = \{0\}$ .

If a commutative algebra  $\mathcal{A}$  has a nondegenerate invariant form  $F$ , then the associated quadratic equation  $\dot{X} = X^2$  in  $\mathcal{A}$  turns out to be a gradient equation with potential function  $\phi(X) = \frac{1}{3}F(X, X^2)$ , that is,  $X^2 = (\nabla\phi)(X)$  for all  $X \in \mathcal{A}$ . Conversely, if the vector field  $X \mapsto X^2$  has a potential function  $\phi : \mathcal{A} \rightarrow \mathbb{K}$  satisfying  $X^2 = (\nabla\phi)(X)$  for all  $X \in \mathcal{A}$ , then the bilinear form  $F : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{K}$  defined by

$$F(X, Y) = \frac{1}{2}\phi^{(2)}(0)(X, Y)$$

is invariant and nondegenerate. (Note that we are using the term “gradient” in a broad sense, for we are not requiring that the form  $F$  be positive definite. Even in the nondegenerate case, one still has the property that trajectories of the differential equation cross the quadrics  $F(X, X) = 0$  orthogonally relative to  $F$  itself.) For further details, see Walcher [15], p.80ff.

Let  $\mathfrak{g}$  be a Lie algebra, and let  $\rho : \mathfrak{g} \rightarrow gl(V)$  be a representation of  $\mathfrak{g}$  as a Lie algebra of linear transformations on some finite-dimensional vector space  $V$ . The *trace form* of  $\mathfrak{g}$  induced by  $\rho$  is the bilinear form  $B_\rho : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$  defined by

$$(7.1) \quad B_\rho(x, y) = \text{tr}(\rho(x)\rho(y))$$

for  $x, y \in \mathfrak{g}$  where  $\text{tr}$  denotes the trace. Using the fact that  $\rho$  is a representation, it is easy to show that  $B_\rho$  is invariant. The most important trace form on a Lie algebra is the one induced by the adjoint representation, which is called the *Killing form*  $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$ , and is given by

$$(7.2) \quad \kappa(x, y) = \text{tr}((\text{ad } x)(\text{ad } y))$$

for  $x, y \in \mathfrak{g}$ . The Killing form measures the structure of  $\mathfrak{g}$  in the following sense:  $\mathfrak{g}$  is semisimple if and only if  $\kappa$  is nondegenerate [5] [13].

We now introduce a related bilinear form on the Nahm algebra  $A(\mathfrak{g})$  which will turn out to measure the structure of  $A(\mathfrak{g})$  in a similar way.

**Definition 7.1.** Let  $\mathfrak{g}$  be a Lie algebra, and let  $\rho : \mathfrak{g} \rightarrow gl(V)$  be a representation of  $\mathfrak{g}$ . The *induced trace form*  $C_\rho : A(\mathfrak{g}) \times A(\mathfrak{g}) \rightarrow \mathbb{K}$  on the associated Nahm algebra  $A(\mathfrak{g})$  is defined by

$$(7.3) \quad C_\rho(X, Y) = \text{tr}(L_\rho(X)L_\rho(Y))$$

for  $X, Y \in A$ . In case  $\rho = \text{ad}$ , the adjoint representation, then we simply write  $C \equiv C_{\text{ad}}$ , and we refer to  $C$  as the *standard form* on  $A(\mathfrak{g})$ .

(See (1.6) for the definition of the operator  $L_\rho(X)$ .)

The induced trace form has an immediate characterization in terms of the trace form of  $\mathfrak{g}$ .

**Theorem 7.2.** For  $X, Y \in A$ ,

$$(7.4) \quad C_\rho(X, Y) = -\frac{1}{2} (B_\rho(x_1, y_1) + B_\rho(x_2, y_2) + B_\rho(x_3, y_3)).$$

*Proof.* Compute  $L_\rho(X)L_\rho(Y)$  using (1.6) and take the trace. ■

*Remark 7.3.* Recall that in addition to having the structure of a Nahm algebra,  $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$  is also a Lie algebra with the component-wise bracket. Since any representation  $\rho : \mathfrak{g} \rightarrow gl(V)$  trivially lifts to a representation  $\hat{\rho} : \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \rightarrow gl(V \times V \times V)$ , trace forms  $B_\rho$  on  $\mathfrak{g}$  induce trace forms  $B_{\hat{\rho}}$  on  $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$  by

$$B_{\hat{\rho}}(X, Y) = B_\rho(x_1, y_1) + B_\rho(x_2, y_2) + B_\rho(x_3, y_3)$$

for  $X, Y \in \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ . We thus have the following relationship between the induced trace form  $C_\rho$  on the Nahm algebra  $A(\mathfrak{g})$  and the induced trace form  $B_{\hat{\rho}}$  on the Lie algebra  $\mathfrak{g} \times \mathfrak{g} \times \mathfrak{g}$ :

$$C_\rho(X, Y) = -\frac{1}{2} B_{\hat{\rho}}(X, Y)$$

for  $X, Y \in A(\mathfrak{g})$ .

**Theorem 7.4.** The induced trace form  $C_\rho : A \times A \rightarrow \mathbb{K}$  is invariant, i.e.,  $C_\rho(XY, Z) = C_\rho(X, YZ)$  for all  $X, Y, Z \in A$ .

*Proof.* Let  $X, Y, Z \in A$  be given. Identify indices modulo 3:  $x_i = x_{i+3}$ , etc. Using (7.4) and the invariance of the trace form  $B_\rho : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$ , we compute

$$\begin{aligned} -2C_\rho(XY, Z) &= \sum_{i=1}^3 B_\rho \left( \frac{1}{2} ([x_{i+1}, y_{i+2}] + [x_{i+2}, y_{i+1}]), z_i \right) \\ &= \sum_{i=1}^3 B_\rho \left( x_{i+1}, \frac{1}{2} [y_{i+2}, z_i] \right) + \sum_{i=1}^3 B_\rho \left( x_{i+2}, \frac{1}{2} [y_{i+1}, z_i] \right). \end{aligned}$$

Since we are summing over all terms, we may reindex and combine them to obtain

$$\begin{aligned} -2C_\rho(XY, Z) &= \sum_{i=1}^3 B_\rho \left( x_i, \frac{1}{2} [y_{i+1}, z_{i+2}] \right) + \sum_{i=1}^3 B_\rho \left( x_i, \frac{1}{2} [y_{i+2}, z_{i+1}] \right) \\ &= \sum_{i=1}^3 B_\rho \left( x_i, \frac{1}{2} ([y_{i+1}, z_{i+2}] + [y_{i+2}, z_{i+1}]) \right) \\ &= -2C_\rho(X, YZ). \end{aligned}$$

This completes the proof. ■

For a Nahm algebra  $A(\mathfrak{g})$ , the radical of an induced trace form is, of course, related to the radical of the corresponding trace form on  $\mathfrak{g}$ .

**Theorem 7.5.**  $\text{rad } C_\rho = \text{rad } B_\rho \times \text{rad } B_\rho \times \text{rad } B_\rho$ .

*Proof.* Fix  $Y = (y_1, y_2, y_3)^T \in \text{rad } C_\rho$ . For all  $x \in \mathfrak{g}$ ,

$$B_\rho(x, y_1) = C_\rho \left( \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \right) = 0,$$

which shows that  $y_1 \in \text{rad } B_\rho$ , and similar computations show  $y_2, y_3 \in \text{rad } B_\rho$ . The reverse inclusion is clear. ■

The following corollaries are immediate.

**Corollary 7.6.** *An induced form  $C_\rho : A \times A \rightarrow \mathbb{K}$  is nondegenerate if and only if the trace form  $B_\rho : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$  is nondegenerate.*

**Corollary 7.7.** *The following are equivalent.*

1. *The standard form  $C : A \times A \rightarrow \mathbb{K}$  is nondegenerate.*
2. *The Killing form  $\kappa : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{K}$  is nondegenerate.*
3.  *$\mathfrak{g}$  is semisimple.*
4.  *$A(\mathfrak{g})$  is semisimple.*

Recall that a Lie algebra  $\mathfrak{g}$  is said to be *compact* if its associated (connected) Lie group is compact. A semisimple Lie algebra is compact if and only if its Killing form is negative definite.

**Definition 7.8.** A Nahm algebra  $A(\mathfrak{g})$  is said to be *compact* if its underlying Lie algebra  $\mathfrak{g}$  is compact.

**Theorem 7.9.** *A semisimple Nahm algebra is compact if and only if its standard form is positive definite.*

*Proof.* This is immediate from (7.4). ■

Recalling our earlier discussion of gradients, it follows that the Nahm equations (1.1)-(1.3) in a compact semisimple Nahm algebra form a gradient system in the traditional sense.

*Remark 7.10.* Recall the diagonal subalgebra  $\Delta(\mathfrak{g})$  of a Nahm algebra  $A(\mathfrak{g})$ . For  $\Delta(x) \in \Delta(\mathfrak{g})$ ,  $Y \in A(\mathfrak{g})$ , we have from (7.4)

$$C(\Delta(x), Y) = -\frac{1}{2}B_\rho(x, y_1 + y_2 + y_3).$$

Thus the orthogonal complement of  $\Delta(\mathfrak{g})$  relative to the standard form is the subspace

$$W_{\text{rad}}(\mathfrak{g}) = \{Y \in A(\mathfrak{g}) : y_1 + y_2 + y_3 \in \text{rad } \kappa\}.$$

The intersection of this subspace with the diagonal subalgebra is

$$\Delta(\mathfrak{g}) \cap W_{\text{rad}}(\mathfrak{g}) = \{\Delta(x) \in \Delta(\mathfrak{g}) : x \in \text{rad } \kappa\}.$$

This is simply a copy of  $\text{rad } \kappa$  itself. In particular, we see from Definition 3.3 that

$$W_{\text{rad}}(\mathfrak{g}) = W(\mathfrak{g}) \text{ if and only if } \text{rad } \kappa = \{0\},$$

that is, if and only if  $\kappa$  and  $C$  are nondegenerate (Corollary 7.7).

## 8. DERIVATIONS

A *derivation* of an algebra  $\mathcal{A}$  is a linear transformation  $D : \mathcal{A} \rightarrow \mathcal{A}$  satisfying  $D(XY) = (DX)Y + X(DY)$  for all  $X, Y \in \mathcal{A}$ . Let  $\text{Der}(\mathcal{A})$  denote the space of all derivations of  $\mathcal{A}$ ; this is a Lie subalgebra of  $gl(\mathcal{A})$ .

If  $\mathcal{A}$  is commutative, then derivations of  $\mathcal{A}$  are linear infinitesimal symmetries of the quadratic differential equation  $\dot{X} = X^2$  in  $\mathcal{A}$ . For  $D \in \text{Der}(\mathcal{A})$ ,  $DX(t; P) = \nabla X(t; P) \cdot DP$ , where  $\nabla$  represents the derivative with respect to the  $\mathcal{A}$ -variables. If  $D \in \text{Der}(\mathcal{A})$  and  $P \in \mathcal{A}$  are such that  $DP = P^2$ , then  $X(t) = e^{tD}P$  turns out to be the unique solution with initial value  $P$ . For more details, see Walcher [15], Kinyon and Sagle [6] [7] [8].

Now for  $1 \leq i < j \leq 3$ , let  $E_{ij} = \mathbf{e}_i \mathbf{e}_j^t - \mathbf{e}_j \mathbf{e}_i^t$ . Then  $\{E_{ij}\}_{1 \leq i < j \leq 3}$  is a basis for the Lie algebra  $so(3, \mathbb{K})$  of  $3 \times 3$  skew-symmetric matrices.

**Theorem 8.1.**  *$so(3, \mathbb{K})$  is a Lie subalgebra of  $\text{Der}(A(\mathfrak{g}))$ .*

*Proof.* We will show that  $E_{12}$  is a derivation. That  $E_{13}$  and  $E_{23}$  are derivations follow similarly. For  $X = (x_1, x_2, x_3)^T, Y = (y_1, y_2, y_3)^T \in A(\mathfrak{g})$ , we compute

$$\begin{aligned} E_{12}(XY) &= \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \left( \frac{1}{2} \begin{pmatrix} [x_2, y_3] + [y_2, x_3] \\ [x_3, y_1] + [y_3, x_1] \\ [x_1, y_2] + [y_1, x_2] \end{pmatrix} \right) \\ (8.1) \quad &= \frac{1}{2} \begin{pmatrix} [x_3, y_1] + [y_3, x_1] \\ -[x_2, y_3] - [y_2, x_3] \\ 0 \end{pmatrix}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} (E_{12}X)Y + X(E_{12}Y) &= \begin{pmatrix} x_2 \\ -x_1 \\ 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \begin{pmatrix} y_2 \\ -y_1 \\ 0 \end{pmatrix} \\ (8.2) \quad &= \frac{1}{2} \begin{pmatrix} -[x_1, y_3] - [y_1, x_3] \\ [y_3, x_2] + [x_3, y_2] \\ [x_2, y_2] - [y_1, x_1] - [x_1, y_1] + [y_2, x_2] \end{pmatrix}. \end{aligned}$$

Comparing (8.1) and (8.2), and using the skew-symmetry of the Lie bracket, the result follows. ■

Next we identify another subalgebra of  $\text{Der}(A(\mathfrak{g}))$ . Observe that the mapping  $\text{diag}(\text{ad}(\cdot)) : \mathfrak{g} \rightarrow gl(\mathcal{A})$  is an isomorphic copy of the adjoint representation of  $\mathfrak{g}$ . Let

$$(8.3) \quad \text{diag}(\text{ad}(\mathfrak{g})) = \{\text{diag}(\text{ad } x) : x \in \mathfrak{g}\}.$$

This is an isomorphic copy of  $\text{ad}(\mathfrak{g})$ , and, if the adjoint representation is faithful, of  $\mathfrak{g}$  itself.

**Theorem 8.2.**  *$\text{diag}(\text{ad}(\mathfrak{g}))$  is a Lie subalgebra of  $\text{Der}(A(\mathfrak{g}))$ .*

*Proof.* The identity

$$\text{diag}(\text{ad } x)(YZ) = (\text{diag}(\text{ad } x)Y)Z + Y(\text{diag}(\text{ad } x)Z)$$

is an easy consequence of the Jacobi identity in  $\mathfrak{g}$ . ■

Obviously  $\text{diag}(\text{ad}(\mathfrak{g})) \cap \mathfrak{so}(3, \mathbb{K}) = \{0\}$ , and thus  $\text{Der}(A(\mathfrak{g}))$  contains  $\text{diag}(\text{ad}(\mathfrak{g})) \oplus \mathfrak{so}(3, \mathbb{K})$  as a direct sum of vector spaces. In addition, for  $x \in \mathfrak{g}$  and  $M \in \mathfrak{so}(3, \mathbb{K})$ , a direct calculation yields

$$(8.4) \quad [\text{diag}(\text{ad } x), M] = 0.$$

Thus  $\text{diag}(\text{ad}(\mathfrak{g})) \oplus \mathfrak{so}(3, \mathbb{K})$  is also an internal direct sum of Lie subalgebras of  $\text{Der}(A(\mathfrak{g}))$ .

The rest of this section will be devoted to proving the following result.

**Theorem 8.3.** *Let  $A(\mathfrak{g})$  be a simple Nahm algebra. Then*

$$\text{Der}(A(\mathfrak{g})) = \text{diag}(\text{ad}(\mathfrak{g})) \oplus \mathfrak{so}(3, \mathbb{K}).$$

Note that the result will turn out to hold for both  $\mathbb{K} = \mathbb{C}$  and  $\mathbb{K} = \mathbb{R}$ .

In the course of the discussion that follows, we will have frequent occasion to use the equivalence of the simplicity of  $A(\mathfrak{g})$  with the simplicity of  $\mathfrak{g}$  (Theorem 5.1) without explicitly mentioning it.

Recall the left multiplication operator  $L(X) : A(\mathfrak{g}) \rightarrow A(\mathfrak{g})$  given by (1.5). We will denote the identity operator in  $\mathfrak{g}$  or  $A(\mathfrak{g})$  by  $I$ , and let the context clarify which is meant. We begin with a version of Schur's Lemma for complex, simple Nahm algebras.

**Lemma 8.4.** *Let  $A(\mathfrak{g})$  be a complex, simple Nahm algebra, and let  $T \in \text{gl}(A(\mathfrak{g}))$  satisfy  $T \circ L(X) = L(X) \circ T$  for all  $X \in A(\mathfrak{g})$ . Then there exists  $\lambda \in \mathbb{C}$  such that  $T = \lambda I$ .*

*Proof.* For  $x \in \mathfrak{g}$ , let  $X = (x, 0, 0)^t$ . Multiplying matrices, we find that the equation  $T \circ L(X) = L(X) \circ T$  is equivalent to the equations

$$\begin{aligned} -T_{12} \circ \text{ad } x &= T_{13} \circ \text{ad } x = 0 \\ -\text{ad } x \circ T_{31} &= \text{ad } x \circ T_{21} = 0 \\ T_{23} \circ \text{ad } x &= -\text{ad } x \circ T_{32} \\ -T_{23} \circ \text{ad } x &= -\text{ad } x \circ T_{33} \\ T_{33} \circ \text{ad } x &= \text{ad } x \circ T_{22} \\ -T_{32} \circ \text{ad } x &= \text{ad } x \circ T_{23}. \end{aligned}$$

If we similarly let  $X = (0, x, 0)^t$  and  $X = (0, 0, x)^t$ , and consider the corresponding matrix equations, then by matching matrix entries, we finally obtain the following system of equations in  $\mathfrak{g}$ :

$$(8.5) \quad T_{ij} \circ \text{ad } x = \text{ad } x \circ T_{ij} = 0$$

$$(8.6) \quad T_{ii} \circ \text{ad } x = \text{ad } x \circ T_{jj}$$



for  $i \neq j$ . By Schur's Lemma, (8.5) implies  $T_{ij} = 0$  for  $i \neq j$ . Identifying indices modulo 3, (8.6) implies

$$T_{ii} \circ \text{ad } x = \text{ad } x \circ T_{i+1,i+1} = T_{i+2,i+2} \circ \text{ad } x = \text{ad } x \circ T_{ii}$$

for  $i = 1, 2, 3$ . By Schur's Lemma, for  $i = 1, 2, 3$ , there exists  $\lambda_i \in \mathbb{C}$  such that  $T_{ii} = \lambda_i I$ . But then (8.6) implies that  $\lambda_1 = \lambda_2 = \lambda_3$ . This completes the proof. ■

For a Nahm algebra  $A(\mathfrak{g})$ , recall the standard form  $C : A(\mathfrak{g}) \times A(\mathfrak{g}) \rightarrow \mathbb{K}$  given by (7.3) or, equivalently, (7.4) where  $\rho$  is the adjoint representation of  $\mathfrak{g}$ . Assume  $\mathfrak{g}$  is semisimple so that  $C$  is nondegenerate (Corollary 7.7). For a linear transformation  $T : A(\mathfrak{g}) \rightarrow A(\mathfrak{g})$ , let  $T^c : A(\mathfrak{g}) \rightarrow A(\mathfrak{g})$  denote the  $C$ -transpose of  $T$  defined by

$$(8.7) \quad C(T^c X, Y) = C(X, TY)$$

for all  $X, Y \in A(\mathfrak{g})$ .

**Lemma 8.5.** *Let  $A(\mathfrak{g})$  be a complex, simple Nahm algebra, and let  $T \in \text{Der}(A(\mathfrak{g}))$ . Then there exists  $\lambda \in \mathbb{C}$  such that  $T + T^c = \lambda I$ .*

*Proof.* For  $X, Y, Z \in A(\mathfrak{g})$ , we compute

$$\begin{aligned} C(X(T^c Y), Z) &= C(T^c Y, XZ) \\ &= C(Y, T(XZ)) \\ &= C(Y, (TX)Z + X(TZ)) \\ &= C(Y, (TX)Z) + C(Y, X(TZ)) \\ &= C(Y(TX), Z) + C(YX, TZ) \\ &= C(Y(TX), Z) + C(T^c(YX), Z) \\ &= C(Y(TX) + T^c(YX), Z) \end{aligned}$$

using the invariance and bilinearity of the standard form, (8.7), and  $T \in \text{Der}(A(\mathfrak{g}))$ . Since  $C$  is nondegenerate (Corollary 7.7),

$$(8.8) \quad X(T^c Y) = Y(TX) + T^c(YX)$$

for all  $X, Y \in A(\mathfrak{g})$ . On the other hand, since  $T$  is a derivation,

$$(8.9) \quad X(TY) = -Y(TX) + T(XY)$$

for all  $X, Y \in A(\mathfrak{g})$ . Adding (8.8) and (8.9), we obtain

$$L(X)(T + T^c)Y = (T + T^c)L(X)Y$$

for all  $X, Y \in A(\mathfrak{g})$ . By Lemma 8.4, there exists  $\lambda \in \mathbb{C}$  such that  $T + T^c = \lambda I$ . ■

It is easy to see that  $\text{diag}(\text{ad } x)$  is  $C$ -skew symmetric for all  $x \in \mathfrak{g}$ , and any matrix in  $so(3, \mathbb{C})$  is  $C$ -skew symmetric as a linear transformation on  $A$ . Thus once Theorem 8.3 is established, it will follow that the conclusion of Lemma 8.5 can be strengthened to the assertion that  $T$  is  $C$ -skew symmetric.

The conclusion of the lemma as it is presently stated will be used in the proof of Theorem 8.3.

Let  $A(\mathfrak{g})$  be a Nahm algebra and let  $T \in gl(A(\mathfrak{g}))$  have the usual block matrix representation  $T = [T_{ij}]$ . Define linear transformations  $T_{\text{diag}}, T_{\text{off}} \in gl(A(\mathfrak{g}))$  by

$$(8.10) \quad T_{\text{diag}} = \begin{pmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{pmatrix}$$

and

$$(8.11) \quad T_{\text{off}} = \begin{pmatrix} 0 & T_{12} & T_{13} \\ T_{21} & 0 & T_{23} \\ T_{31} & T_{32} & 0 \end{pmatrix}.$$

For  $X \in A(\mathfrak{g})$ , we compute

$$(8.12) \quad T(X^2) = \begin{pmatrix} T_{11}[x_2, x_3] + T_{12}[x_3, x_1] + T_{13}[x_1, x_2] \\ T_{21}[x_2, x_3] + T_{22}[x_3, x_1] + T_{23}[x_1, x_2] \\ T_{31}[x_2, x_3] + T_{32}[x_3, x_1] + T_{33}[x_1, x_2] \end{pmatrix}$$

and  $2(TX)X =$

$$(8.13) \quad \begin{pmatrix} [T_{21}x_1 + T_{22}x_2 + T_{23}x_3, x_3] + [x_2, T_{31}x_1 + T_{32}x_2 + T_{33}x_3] \\ [T_{31}x_1 + T_{32}x_2 + T_{33}x_3, x_1] + [x_3, T_{11}x_1 + T_{12}x_2 + T_{13}x_3] \\ [T_{11}x_1 + T_{12}x_2 + T_{13}x_3, x_2] + [x_1, T_{21}x_1 + T_{22}x_2 + T_{23}x_3] \end{pmatrix}.$$

**Lemma 8.6.** *Let  $A(\mathfrak{g})$  be a Nahm algebra and let  $T \in \text{Der}(A(\mathfrak{g}))$ . Then  $T_{\text{diag}} \in \text{Der}(A(\mathfrak{g}))$  and  $T_{\text{off}} \in \text{Der}(A(\mathfrak{g}))$ .*

*Proof.* Since  $T$  is a derivation, (8.12) and (8.13) are equal. Take  $x_2 = x_3 = 0$ ,  $x_1 = x$  in (8.12) and (8.13) and match the entries. This gives  $[T_{31}x, x] = 0$  and  $[x, T_{21}x] = 0$  for all  $x \in \mathfrak{g}$ . By similar arguments, we obtain

$$(8.14) \quad [T_{ij}x, x] = 0$$

for all  $x \in \mathfrak{g}$  where  $i \neq j$ . Linearizing (8.14) and rearranging, we have

$$(8.15) \quad [T_{ij}x, y] = [x, T_{ij}y]$$

for all  $x, y \in \mathfrak{g}$  where  $i \neq j$ . Take  $x_1 = 0$  in (8.12) and (8.13), simplify using (8.14), and match the first entries. This gives

$$(8.16) \quad T_{11}[x_2, x_3] = [T_{22}x_2, x_3] + [x_2, T_{33}x_3]$$

for all  $x_2, x_3 \in \mathfrak{g}$ . Successively taking  $x_2 = 0$  and  $x_3 = 0$  give the equations

$$(8.17) \quad T_{22}[x_3, x_1] = [T_{33}x_3, x_1] + [x_3, T_{11}x_1]$$

$$(8.18) \quad T_{33}[x_1, x_2] = [T_{11}x_1, x_2] + [x_1, T_{22}x_2]$$

for all  $x_1, x_2, x_3 \in \mathfrak{g}$ . Taken together, (8.16), (8.17) and (8.18) imply

$$T_{\text{diag}}(X^2) = 2X(T_{\text{diag}}X)$$

for all  $X \in A(\mathfrak{g})$ , i.e.,  $T_{\text{diag}}$  is a derivation of  $A(\mathfrak{g})$ . Since  $T_{\text{off}} = T - T_{\text{diag}}$ ,  $T_{\text{off}}$  is also a derivation. ■

**Lemma 8.7.** *Let  $A(\mathfrak{g})$  be a complex, simple Nahm algebra, and let  $T \in \text{Der}(A(\mathfrak{g}))$  be given. Assume  $T = T_{\text{off}}$ . Then the action of  $T$  on  $A(\mathfrak{g})$  is given by the action of a matrix in  $\text{so}(3, \mathbb{K})$ .*

*Proof.* Using (8.14), the equality of (8.12) and (8.13) simplifies to

$$(8.19) \quad \begin{pmatrix} T_{12}[x_3, x_1] + T_{13}[x_1, x_2] \\ T_{21}[x_2, x_3] + T_{23}[x_1, x_2] \\ T_{31}[x_2, x_3] + T_{32}[x_3, x_1] \end{pmatrix} = \begin{pmatrix} [T_{21}x_1, x_3] + [x_2, T_{31}x_1] \\ [T_{32}x_2, x_1] + [x_3, T_{12}x_2] \\ [T_{13}x_3, x_2] + [x_1, T_{23}x_3] \end{pmatrix}$$

for all  $x_1, x_2, x_3 \in \mathfrak{g}$ . Set  $x_3 = 0$  in (8.19) and match entries. Using (8.15), this gives

$$\begin{aligned} T_{13}[x_1, x_2] &= [x_2, T_{31}x_1] = -[x_1, T_{31}x_2] \\ T_{23}[x_1, x_2] &= [T_{32}x_2, x_1] = -[x_1, T_{32}x_2] \end{aligned}$$

for all  $x_1, x_2 \in \mathfrak{g}$ . Similar calculations give

$$(8.20) \quad T_{ij}[x, y] = -[x, T_{ji}y]$$

for all  $x, y \in \mathfrak{g}$  where  $i \neq j$ . Iterating (8.20), we have

$$\begin{aligned} T_{ij}[x, [y, z]] &= -[x, T_{ji}[y, z]] \\ &= [x, [y, T_{ij}z]] \end{aligned}$$

for all  $x, y, z \in \mathfrak{g}$ . Thus

$$(8.21) \quad T_{ij} \circ \text{ad } x \circ \text{ad } y = \text{ad } x \circ \text{ad } y \circ T_{ij}$$

for all  $x, y \in \mathfrak{g}$ . Reversing the roles of  $x$  and  $y$  in (8.21) and subtracting the resulting equation from (8.21), we obtain

$$(8.22) \quad T_{ij} \circ \text{ad } [x, y] = \text{ad } [x, y] \circ T_{ij}$$

for all  $x, y \in \mathfrak{g}$  since  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{g}$  is a representation. Since  $\mathfrak{g}$  is simple, we have  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$  and thus (8.22) implies

$$T_{ij} \circ \text{ad } x = \text{ad } x \circ T_{ij}$$

for all  $x \in \mathfrak{g}$ . By Schur's Lemma, there exists  $\lambda_{ij} \in \mathbb{C}$  ( $i \neq j$ ) such that  $T_{ij} = \lambda_{ij}I$ . If we set  $\lambda_{ii} = 0$  for  $i = 1, 2, 3$ , then the action of  $T$  on  $A$  is given by the action of the matrix  $\Lambda = [\lambda_{ij}]$ . What remains is to show that  $\Lambda \in \text{so}(3, \mathbb{C})$ , and for this purpose we will use Lemma 8.5. Let  $X, Y \in A(\mathfrak{g})$  be given. Using (8.7) and (7.4), we compute

$$\begin{aligned} C(T^c X, Y) &= C(X, TY) \\ &= -\frac{1}{2} \sum_{i=1}^3 B \left( x_i, \sum_{j=1}^3 \lambda_{ij} y_j \right) \\ &= -\frac{1}{2} \sum_{j=1}^3 B \left( \sum_{i=1}^3 \lambda_{ij} x_i, y_j \right) \\ &= C(\Lambda^t X, Y) \end{aligned}$$

where  $\Lambda^t$  is the usual transpose of the matrix  $\Lambda$ . Since  $C$  is nondegenerate, the action of  $T^c$  on  $A(\mathfrak{g})$  is given by the action of the matrix  $\Lambda^t$ . By Lemma 8.5, we have

$$\Lambda + \Lambda^t = \mu I$$

for some  $\mu \in \mathbb{C}$ . But the diagonal entries of  $\Lambda$ , and hence  $\Lambda^t$ , are all zero, and thus  $\mu = 0$ . It follows that  $\Lambda \in so(3, \mathbb{C})$ . This completes the proof. ■

**Lemma 8.8.** *Let  $A(\mathfrak{g})$  be a complex, simple Nahm algebra, and let  $T \in \text{Der}(A(\mathfrak{g}))$  be given. Assume  $T = T_{\text{diag}}$ . Then there exists  $x \in \mathfrak{g}$  such that  $T = \text{diag}(\text{ad } x)$ .*

*Proof.* The equality of (8.12) and (8.13) gives the following system of equations

$$(8.23) \quad T_{11}[x_2, x_3] = [T_{22}x_2, x_3] + [x_2, T_{33}x_3]$$

$$(8.24) \quad T_{22}[x_3, x_1] = [T_{33}x_3, x_1] + [x_3, T_{11}x_1]$$

$$(8.25) \quad T_{33}[x_1, x_2] = [T_{11}x_1, x_2] + [x_1, T_{22}x_2]$$

for all  $x_1, x_2, x_3 \in \mathfrak{g}$ . Set  $x = x_1 = x_2$  and  $y = x_3$ , and add (8.23) and (8.24) to obtain

$$(8.26) \quad \begin{aligned} (T_{11} - T_{22})[x, y] &= [T_{22}x, y] + [x, T_{33}y] + [T_{33}y, x] + [y, T_{11}x] \\ &= [y, (T_{11} - T_{22})x] \end{aligned}$$

for all  $x, y \in \mathfrak{g}$ . Iterating (8.26), we obtain

$$\begin{aligned} (T_{11} - T_{22})[z, [y, x]] &= (T_{11} - T_{22})[[x, y], z] \\ &= [z, (T_{11} - T_{22})[x, y]] \\ &= [z, [y, (T_{11} - T_{22})x]] \end{aligned}$$

for all  $x, y, z \in \mathfrak{g}$ . Thus

$$(8.27) \quad (T_{11} - T_{22}) \circ \text{ad } z \circ \text{ad } y = \text{ad } z \circ \text{ad } y \circ (T_{11} - T_{22})$$

for all  $y, z \in \mathfrak{g}$ . Exchanging  $y$  and  $z$  in (8.27) and subtracting the resulting equation from (8.27) gives

$$(T_{11} - T_{22}) \circ \text{ad } [z, y] = \text{ad } [z, y] \circ (T_{11} - T_{22})$$

for all  $y, z \in \mathfrak{g}$  since  $\text{ad} : \mathfrak{g} \rightarrow \mathfrak{g}$  is a representation. Now  $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$  because  $\mathfrak{g}$  is simple, and thus  $(T_{11} - T_{22}) \circ \text{ad } z = \text{ad } z \circ (T_{11} - T_{22})$  for all  $z \in \mathfrak{g}$ . By Schur's Lemma, there exists  $\lambda_{12} \in \mathbb{C}$  such that  $T_{11} - T_{22} = \lambda_{12}I$ . Applying this to (8.26), we obtain

$$\lambda_{12}[x, y] = [y, \lambda_{12}x] = \lambda_{12}[y, x] = -\lambda_{12}[x, y]$$

for all  $x, y \in \mathfrak{g}$ . Thus  $\lambda_{12} = 0$ , and hence  $T_{11} = T_{22}$ . Similar arguments, *mutatis mutandis*, show that  $T_{22} = T_{33}$ . Now (8.23), say, shows that  $T_{ii}$  is a derivation of  $\mathfrak{g}$ . Since  $\mathfrak{g}$  is simple, there exists  $x \in \mathfrak{g}$  such that  $T_{ii} = \text{ad } x$ . It follows that  $T = \text{diag}(\text{ad } x)$  as claimed. ■

Finally, we complete the proof of Theorem 8.3. If  $\mathbb{K} = \mathbb{C}$ , then the result follows from Lemmas 8.6, 8.7, and 8.8. Now suppose  $\mathbb{K} = \mathbb{R}$ . By Theorems 8.2 and 8.1, we have that  $\text{diag}(\text{ad}(\mathfrak{g})) \oplus \text{so}(3, \mathbb{R})$  is a subalgebra of  $\text{Der}(A(\mathfrak{g}))$ . Now the complexification of  $A(\mathfrak{g})$  is simple, and its derivation algebra is the complexification of  $\text{Der}(A(\mathfrak{g}))$ . But the real dimension of a derivation algebra is equal to the complex dimension of its complexification. This gives us the desired result.

## 9. AUTOMORPHISMS

An *automorphism* of an algebra  $\mathcal{A}$  is an invertible linear mapping  $\phi : \mathcal{A} \rightarrow \mathcal{A}$  satisfying  $\phi(XY) = \phi(X)\phi(Y)$  for all  $X, Y \in \mathcal{A}$ . Let  $\text{Aut}(\mathcal{A})$  denote the set of all automorphisms of  $\mathcal{A}$ ; this is a closed (Lie) subgroup of  $GL(\mathcal{A})$ , and  $\text{Der}(\mathcal{A})$  is the Lie algebra of  $\text{Aut}(\mathcal{A})$  (see Sagale and Walde [13]).

If  $\mathcal{A}$  is commutative, then the automorphisms of  $\mathcal{A}$  are the linear symmetries of the quadratic differential equation  $\dot{X} = X^2$  occurring in  $\mathcal{A}$ , that is, they are solution preserving. Let  $X(t) = X(t; P)$  denote the unique solution with initial value  $P \in \mathcal{A}$ , and let  $Y(t) = \phi(X(t))$  for  $\phi \in \text{Aut}(\mathcal{A})$ . Then

$$\dot{Y}(t) = \phi(\dot{X}(t)) = \phi(X^2(t)) = [\phi(X(t))]^2 = Y(t)^2.$$

Since  $Y(0) = \phi P$ , we have that  $Y(t) = Y(t; \phi P)$  is the unique solution with initial value  $\phi P$ . For more on using automorphisms to study quadratic differential equations, see Walcher [15], Kinyon and Sagale [6] [7] [8], Hopkins and Kinyon [3] [4].

Turning to Nahm algebras, we have the following immediate corollary of Theorem 8.1

**Corollary 9.1.**  $SO(3, \mathbb{K}) \leq \text{Aut}(A(\mathfrak{g}))$ .

*Remark 9.2.* If  $\mathcal{A}$  is a commutative algebra with  $\mathbb{Z}_2$ -grading  $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ , then the mapping given by  $X_0 + X_1 \mapsto X_0 - X_1$  is an automorphism of order 2, and conversely, any such automorphism induces a  $\mathbb{Z}_2$ -grading by setting  $\mathcal{A}_0$  and  $\mathcal{A}_1$  equal to the +1- and -1-eigenspaces, respectively. Recall the  $\mathbb{Z}_2$ -grading  $A(\mathfrak{g}) = \Delta(\mathfrak{g}) \oplus W(\mathfrak{g})$  established in Corollary 3.2. Let  $U$  denote the automorphism that defines the grading. Then  $U$  is given by

$$U(X) = P_\Delta(X) - P_W(X)$$

for  $X \in A(\mathfrak{g})$ , where  $P_\Delta$  and  $P_W$  are the projectors defined in (3.8) and (3.9), respectively. Computing  $U(X)$  explicitly in terms of the entries, we find

$$U(X) = \frac{1}{3} \begin{pmatrix} -x_1 + 2x_2 + 2x_3 \\ 2x_1 - x_2 + 2x_3 \\ 2x_1 + 2x_2 - x_3 \end{pmatrix}$$

for all  $X \in A(\mathfrak{g})$ , which implies we can identify  $U$  with a  $3 \times 3$  matrix:

$$(9.1) \quad U = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}.$$

This matrix is orthogonal and has determinant 1, and thus  $U \in SO(3, \mathbb{K})$ . A particular derivation  $G$  such that  $\exp G = U$  is given by

$$G = \frac{\pi}{\sqrt{3}} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

Of course, automorphisms of  $\mathfrak{g}$  induce automorphisms of  $A(\mathfrak{g})$  in the obvious way: for  $\phi \in \text{Aut}(\mathfrak{g})$ , we clearly have  $\text{diag}(\phi) \in \text{Aut}(A(\mathfrak{g}))$ .

**Proposition 9.3.**  $\text{diag}(\text{Aut}(\mathfrak{g})) \leq \text{Aut}(A(\mathfrak{g}))$ .

Now we show that for simple Nahm algebras, Corollary 9.1 and Proposition 9.3 describe all the automorphisms. One particular implication of this is that a Nahm algebra has no outer automorphisms other than those it inherits from its underlying Lie algebra.

**Theorem 9.4.** *Let  $A(\mathfrak{g})$  be a simple Nahm algebra. Then*

$$\text{Aut}(A(\mathfrak{g})) = \text{diag}(\text{Aut}(\mathfrak{g})) \times SO(3, \mathbb{K}).$$

*Proof.* Let  $f \in \text{Aut}(A(\mathfrak{g}))$  be given. Define  $\hat{f} \in \text{gl}(\text{Der}(A(\mathfrak{g})))$  by  $\hat{f}(T) = f \circ T \circ f^{-1}$  for all  $T \in \text{Der}(A(\mathfrak{g}))$ . Then  $\hat{f}$  is an automorphism of  $\text{Der}(A(\mathfrak{g}))$ . Now

$$\text{Aut}(\text{Der}(A(\mathfrak{g}))) = \text{Aut}(\text{diag}(\text{ad}(\mathfrak{g}))) \times SO(3, \mathbb{K}),$$

using Theorem 8.3 and the fact that  $\text{Aut}(\text{so}(3, \mathbb{K})) = SO(3, \mathbb{K})$  [5]. We have  $\text{diag}(\text{ad}(\mathfrak{g})) \cong \text{ad}(\mathfrak{g})$ . Thus there exists  $\hat{\phi} \in \text{Aut}(\text{ad}(\mathfrak{g}))$  such that

$$\hat{f}(\text{diag}(\text{ad } x)) = \text{diag}(\hat{\phi}(\text{ad } x))$$

for all  $x \in \mathfrak{g}$ . Since  $\mathfrak{g}$  is simple,  $\mathfrak{g} \cong \text{ad}(\mathfrak{g})$ . Thus define  $\phi \in \text{Aut}(\mathfrak{g})$  by  $\hat{\phi}(\text{ad } x) = \text{ad } \phi(x)$  for all  $x \in \mathfrak{g}$ . Then  $\hat{\phi}(\text{ad } x) = \phi \circ \text{ad } x \circ \phi^{-1}$  and hence

$$\text{diag}(\hat{\phi}(\text{ad } x)) = \text{diag}(\phi) \circ \text{diag}(\text{ad } x) \circ \text{diag}(\phi^{-1})$$

for all  $x \in \mathfrak{g}$ . Next, there exists  $R \in SO(3, \mathbb{K})$  such that

$$\hat{f}(M) = RMR^{-1}$$

for all  $M \in \text{so}(3, \mathbb{K})$ . Putting this together, we find that

$$f \circ T \circ f^{-1} = \hat{f}(T) = (\text{diag}(\phi) \circ R) \circ T \circ (\text{diag}(\phi) \circ R)^{-1}$$

for all  $T \in \text{Der}(A(\mathfrak{g}))$ , where  $\text{diag}(\phi) \circ R = R \circ \text{diag}(\phi)$ . By Lemma 8.4, it follows that  $f^{-1} \circ \text{diag}(\phi) \circ R = \lambda I$  for some  $\lambda \in \mathbb{K}$ . Since the left side is an automorphism,  $\lambda = 1$ , and thus  $f = \text{diag}(\phi) \circ R$ . This completes the proof. ■

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